

The Helmholtz' decomposition of decreasing and weakly increasing vector fields

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Abstract

Helmholtz' decomposition theorem for vector fields is presented usually with too strong restrictions on the fields. Based on the work of Blumenthal of 1905 it is shown that the decomposition of vector fields is not only possible for asymptotically weakly decreasing vector fields, but even for vector fields, which asymptotically increase sublinearly. Use is made of a regularization of the Green's function and the mathematics of the proof is formulated as simply as possible. We also show a few examples for the decomposition of vector fields including the electric dipole radiation.

Keywords: Helmholtz theorem, vector field, electromagnetic radiation

I. INTRODUCTION

According to the Helmholtz' theorem one can decompose a given vector field $\vec{v}(\vec{x})$ into a sum of two vector fields $\vec{v}_l(\vec{x})$ and $\vec{v}_t(\vec{x})$ where \vec{v}_l is irrotational (curl-free) and \vec{v}_t solenoidal (divergence-free), if the vector field fulfills certain conditions on continuity and asymptotic decrease ($r \rightarrow \infty$). Here \vec{x} is the position vector in three-dimensional space and $r = |\vec{x}|$ its absolute value.

The two parts of the vector field can be expressed as gradient of a scalar potential and curl of a vector potential, respectively. Concerning the validity, the uniqueness of the decomposition and the existence of the respective potentials one finds different conditions.

The fundamental theorem for vector fields is historically based on Helmholtz' work on vortices^{1,2} and therefore also known as Helmholtz' decomposition theorem. For hydrodynamics this theorem is of particular relevance, since the fluid fields of the decomposition have the physical properties of freedom of vorticity and incompressibility, which for each field makes the analysis simpler. Especially for the visualization of vector fields the decomposition theorem is of importance⁴.

Föppl⁵ introduced the decomposition theorem into electrodynamics. He assumed a finite extension of the sources and vortices and therefore assumed a behavior for the corresponding vector field of the form $|\vec{v}| \sim 1/r^2$ for $|\vec{x}| = r \rightarrow \infty$. However, his proof allows less restrictive conditions, namely an asymptotic decay of the field only somewhat stronger than $1/r$. The decomposition theorem can be found in one of these formulations in most textbooks or lecture notes on electrodynamics.

Already in 1905 Otto Blumenthal⁶ proved, that any vector field, that goes to zero asymptotically can be decomposed in a curl-free and a divergence-free part (weak version). His formulation reads as follows⁷:

"Let \vec{v} be a vector, which is in addition to arbitrary many derivatives everywhere finite and continuous and vanishes at infinity with its derivatives; then one can decompose this vector always into two vectors, a curl-free \vec{v}_l and a divergence-free \vec{v}_t , such that

$$\vec{v} = \vec{v}_l(\vec{x}) + \vec{v}_t(\vec{x}). \quad (*)$$

The vectors \vec{v}_l and \vec{v}_t diverge asymptotically weaker than $\ln r$.

In addition one has the following proposition for uniqueness: \vec{v}_l and \vec{v}_t are unique up to an

additive constant vector, because of the given properties.” No further specification for the behavior of the vector field was given.

This formulation was taken over in its essential statements by Sommerfeld⁸ in 1944. He noted further that the fundamental theorem of vector analysis, as he called it, was already proven by Stokes² in 1849 and in a more complete form by Helmholtz’ paper of 1858.

The extension to a decay of $1/r$ and weaker is important for electromagnetic radiation but also for a few configurations in electro- and magnetostatics.

Later on it was shown that the conditions of continuity and differentiability can be weakened^{4,9} and that the theorem can be applied to vector fields behaving according to a certain power law¹⁰. Based on Blumenthal’s method of regularization of the Green’s function Neudert and Wahl¹¹ investigated among other things the asymptotic behavior of a vector field \vec{v} if its sources $\text{div } \vec{v}$ and vortices $\text{curl } \vec{v}$ fulfill some conditions including differentiability and asymptotic decay.

These developments remained to a large extent unnoticed in the physical literature¹² and in mathematical physics¹³. Thus it was necessary to show the validity of the decomposition theorem for electromagnetic radiation fields¹⁴ that decay asymptotically with $1/r$.

First we develop a systematic method, the so called regularization method, which is the basis of Blumenthal’s proof, but is not explicated in its generality and its improvement in order to be applicable to vector fields, which decay asymptotically with a specified power law. Then we reformulated the decomposition theorem including all potentials for such cases. It is shown how the levels of the regularization modifies the validity of the uniqueness for the vector fields depending on their behavior at infinity. The equations necessary to construct the decomposed parts are presented. The necessity of a formulation of a proper asymptotic condition either for the irrotational or solenoidal part is pointed out.

In the next section we apply the decomposition theorem to the electromagnetic dipole radiation making clear why the traditional theorem is applicable and how the different quantities in the theorem are related to electrodynamics.

Then we formulate and proof the extension of the theorem and uniqueness up to a constant vector for sublinearly diverging vector fields. A mathematical example without physical background is given in an Appendix. Such a case might be of interest for physics, if one has sources (circulations) that remain finite even at infinity. In the conclusion we summarize the results.

II. REGULARIZATION METHOD

The solution $\phi_0(\vec{x})$ of the Poisson equation

$$\Delta\phi_0(\vec{x}) = -4\pi\rho(\vec{x}) \quad (1)$$

with the source density $\rho(\vec{x})$ is found by introducing it's Green's function

$$G_0(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}' - \vec{x}|} \quad (2)$$

$$\phi_0(\vec{x}) = \int d^3x' \rho(\vec{x}') G_0(\vec{x}, \vec{x}'). \quad (3)$$

If the solution exists in the whole domain of \mathbb{R}^3 , the integral has to be finite. This is guaranteed by a sufficient decay of the integrand, either by a sufficient strong decay of the source density and/or by a sufficient decrease of the Green's function.

In his work on the Helmholtz' theorem⁶ Blumenthal presented a method to make this solution finite (regularizing the solution) by changing the Green's function of the Poisson equation, without changing the Poisson equation (that means without changing the source density). Thus one can prove the existence of the potential for cases where the source density is less strong decreasing. From this method it becomes clear how a systematic extension of the decomposition theorem is possible.

Introducing an arbitrary point \vec{x}_0 (apart from the condition that $\rho(\vec{x}_0)$ is finite at this point; regularization point) and noting that $G_0(\vec{x}, \vec{x}') = G_0(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0)$, we expand G_0 in a power series in $\vec{x} - \vec{x}_0$

$$G_0(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}' - \vec{x}_0|} + \frac{(\vec{x} - \vec{x}_0) \cdot (\vec{x}' - \vec{x}_0)}{|\vec{x}' - \vec{x}_0|^3} + O\left(\frac{1}{|\vec{x}' - \vec{x}_0|^3}\right). \quad (4)$$

A stronger decrease for large $|\vec{x}'|$ of the Green's function is now reached by subtraction of the corresponding expansion terms. We get the following set of stronger decreasing Green's functions

$$G_1(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = G_0(\vec{x}, \vec{x}') - \frac{1}{|\vec{x}' - \vec{x}_0|} \quad (5)$$

$$G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = G_1(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) - \frac{(\vec{x} - \vec{x}_0) \cdot (\vec{x}' - \vec{x}_0)}{|\vec{x}' - \vec{x}_0|^3}. \quad (6)$$

The asymptotic decrease of these modified Green's functions is as $\sim 1/r'^{1+i}$. For $i \leq 2$ the subtracted terms do not change the source density

$$\Delta G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = -4\pi\delta(\vec{x}' - \vec{x}) \quad \text{for} \quad 0 \leq i \leq 2. \quad (7)$$

But they allow to extend the range of the validity for which the existence of the potential (and the decomposition) can be proven

$$\phi_i(\vec{x}) = \int d^3x' \rho(\vec{x}') G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) \quad \text{and} \quad \Delta\phi_i(\vec{x}) = -4\pi\rho(\vec{x}) \quad \text{for } i \leq 2. \quad (8)$$

The solutions $\phi_i(\vec{x})$ differ only by a (divergence- and curl-free) solution of the Laplace equation, i.e. $\phi_0(\vec{x})$ differs from $\phi_1(\vec{x})$ by a constant value and from $\phi_2(\vec{x})$ by a linear function, both depending on \vec{x}_0 .

Trying to extend the range of validity even further one may subtract the next (third) term in the expansion (4) from G_2 and obtains

$$G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) - \frac{1}{2}((\vec{x} - \vec{x}_0) \cdot \vec{\nabla}')^2 \frac{1}{|\vec{x}' - \vec{x}_0|}. \quad (9)$$

But now G_3 fulfills the Poisson equation

$$\Delta G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = -4\pi[\delta(\vec{x}' - \vec{x}) - \delta(\vec{x}' - \vec{x}_0)] \quad (10)$$

from which it follows, that G_3 leads to a solution of a modified Poisson equation

$$\Delta\phi_3(\vec{x}) = -4\pi[\rho(\vec{x}) - \rho(\vec{x}_0)]. \quad (11)$$

Thus the method described here is not suitable for Green's functions G_i with $i > 2$. This means (as we will see later) that vector fields which increase linearly or even stronger will not be decomposed by the regularization method described here.

Nevertheless one should remark that one can solve the Poisson equation even with G_3 if one subtracts the solution for the inhomogeneity $\rho(\vec{x}_0)$

$$\bar{\phi}_3(\vec{x}) = \int d^3x' \rho(\vec{x}') G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) + 2\pi \frac{\rho(\vec{x}_0)}{3} r^2. \quad (12)$$

We refer to this solution in section V A 4.

The relations

$$\vec{\nabla} G_{i+1}(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = -\vec{\nabla}' G_i(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) \quad \text{for } i \leq 2 \quad (13)$$

can be derived from (5), (6) and (9). They are used a few times, mainly to compute the vector fields \vec{v}_l and \vec{v}_t and to establish relations between them.

In the following we will restrict ourselves to the regularization point $\vec{x}_0 = 0$, because the Green's functions are simpler without loss of generality. In this case the potential is fixed to $\phi(\vec{x} = 0) = 0$. We will keep this choice in the remaining part of the paper as far as possible.

III. THE FUNDAMENTAL THEOREM OF VECTOR ANALYSIS

As already noticed, the formulation of the fundamental theorem rests in its form today on the work of Blumenthal. However there are several reasons not to take the formulations of Blumenthal resp. Sommerfeld literally. For instance the uniqueness of the decomposition into the fields of the sources and vortices, was only shown up to a constant vector. We will formulate the conditions in such a form, that a strict uniqueness of the decomposition is given. Furthermore in the proof, which will be given, the potentials by which the decomposed fields are calculated, are part of the theorem (strong version). It is common in electrodynamics to calculate the physical fields via the introduction of potentials. Moreover since the proof of Blumenthal is somewhat complex and lengthy it is not found in detail in textbooks. Therefore a shorter and more compact proof seems to be useful. Thus we formulate the theorem in the following way:

Let $\vec{v}(\vec{x})$ be an everywhere continuous differentiable vector field with the asymptotic behavior $\lim_{r \rightarrow \infty} v(r) r^\epsilon < \infty$, where $\epsilon > 0$, then the decomposition

$$\vec{v}(\vec{x}) = \vec{v}_l + \vec{v}_t = -\vec{\nabla}\phi(\vec{x}) + \vec{\nabla} \times \vec{A}(\vec{x}) \quad (14)$$

is unique with

$$\phi(\vec{x}) = \frac{1}{4\pi} \int d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) \left(\frac{1}{|\vec{x}' - \vec{x}|} - \frac{1}{r'} \right) \quad (15)$$

$$\vec{A}(\vec{x}) = \frac{1}{4\pi} \int d^3x' (\vec{\nabla}' \times \vec{v}(\vec{x}')) \left(\frac{1}{|\vec{x}' - \vec{x}|} - \frac{1}{r'} \right). \quad (16)$$

Remarks:

- Curl- and divergence-free fields \vec{v}_h can be added to \vec{v}_l if they are subtracted from \vec{v}_t without affecting the boundary conditions of \vec{v} . Such harmonic vector fields are suppressed if one explicitly demands that \vec{v}_l and/or \vec{v}_t vanishes asymptotically and establish a strict uniqueness of the decomposition.
- Usually the potentials $\phi(\vec{x})$ and $\vec{A}(\vec{x})$ are defined with the Green's function G_0 (2). If they are finite, then there is no need for G_1 (5). However if the vector field \vec{v} decays asymptotically as $1/r$ or weaker, one generally has to use the Green's function G_1 as shown in (15) and (16) in order to avoid divergences in the potentials $\phi(\vec{x})$ and $\vec{A}(\vec{x})$.

- As already mentioned in section II, the potentials are fixed to the values $\phi(0) = 0$ and $\vec{A}(0) = 0$ by the choice of the regularization point $\vec{x}_0 = 0$. This choice does not affect the vector fields \vec{v}_l and \vec{v}_t .
- The vector potential \vec{A} by its definition is purely transversal, $\vec{\nabla} \cdot \vec{A} = 0$ (see proof below in App. A).
- We want to stress the point that the decomposition theorem holds for any vector field independent of the type of physical equations the vector field might fulfill. On the other hand if one thinks of the electric field or the magnetic field as examples of the theorem, due to the Maxwell equations these fields are connected although with respect to the decomposition theorem they are independent.

Let us define the source density $\rho(\vec{x})$ and the vortex density $\vec{j}(\vec{x})$ as

$$\rho(\vec{x}) = \frac{\vec{\nabla} \cdot \vec{v}(\vec{x})}{4\pi} \qquad \vec{j}(\vec{x}) = \frac{\vec{\nabla} \times \vec{v}(\vec{x})}{4\pi}, \quad (17)$$

then the decomposition of the corresponding vector field in its irrotational (curl-free) and solenoidal (divergence-free) parts leads to the result, that

$$\vec{\nabla} \cdot \vec{v}_l(\vec{x}) = 4\pi\rho(\vec{x}) \qquad \text{and} \qquad \vec{\nabla} \times \vec{v}_l(\vec{x}) = 0 \quad (18)$$

$$\vec{\nabla} \times \vec{v}_t(\vec{x}) = 4\pi\vec{j}(\vec{x}) \qquad \text{and} \qquad \vec{\nabla} \cdot \vec{v}_t(\vec{x}) = 0. \quad (19)$$

1. Proof of the fundamental theorem

First we show the existence of the scalar potential. If the finiteness of the integral (15) is proven, one gets the field \vec{v}_l by calculating the gradient of ϕ . For this it is required that the integration and differentiation interchange. Then one can show that $\vec{\nabla} \times \vec{v}_l = 0$ and $\vec{\nabla} \cdot \vec{v}_l = \vec{\nabla} \cdot \vec{v}$.

Subsequently one proceeds quite similarly for the vortex field by showing the existence of (16) first, then calculating \vec{v}_t and proving its properties $\vec{\nabla} \times \vec{v}_t = \vec{\nabla} \times \vec{v}$ and $\vec{\nabla} \cdot \vec{v}_t = 0$. Finally we check that the sum $\vec{v}_l + \vec{v}_t = \vec{v}$.

If we show that the integral (15) exists and is finite, then the longitudinal part \vec{v}_l can be determined. We note, that the singularities at \vec{x} and at zero do not lead to a diverging

contribution to the integral, because of the antisymmetry of the integrand around the singularity. More important is the asymptotic behavior of the integral for $r' \rightarrow \infty$. We integrate then over the surface of a larger sphere with radius R . Now we have to take into account the regularization term (5). Since no assumptions have been made on the asymptotic behavior of the sources $\rho = \vec{\nabla} \cdot \vec{v}/4\pi$ but only on \vec{v} , we perform a partial integration. This allows us to prove the convergence from the behavior of the vector field \vec{v} alone. Integrating over the volume of the sphere leads to

$$\begin{aligned} \phi(\vec{x}) &\stackrel{R \gg r}{=} \frac{1}{4\pi} \int_{S_R} d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) G_1(\vec{x}, \vec{x}') \\ &= \frac{1}{4\pi} \oint_{\partial S_R} d\vec{f}' \cdot \underbrace{\vec{v}(\vec{x}')}_{\sim 1/R^\epsilon} \underbrace{G_1(\vec{x}, \vec{x}')}_{\sim 1/R^2} - \frac{1}{4\pi} \int_{S_R} d^3x' \vec{v}(\vec{x}') \cdot \vec{\nabla}' G_1(\vec{x}, \vec{x}'). \end{aligned} \quad (20)$$

The radius R can be chosen in such a way, that the field becomes small. Then the surface integral vanishes as $1/R^\epsilon$ and it remains to show convergence of the volume integral.

In order to achieve this we separate the volume of integration into an inner volume of a sphere S_R with radius $R \gg r$ and the outer domain $r' \geq R$

$$\phi(\vec{x}) \stackrel{R \gg r}{=} -\frac{1}{4\pi} \int_{S_R} d^3x' (\vec{v}(\vec{x}') \cdot \vec{\nabla}') G_1(\vec{x}, \vec{x}') + \phi_a(\vec{x}). \quad (21)$$

The contribution of the outer domain to the potential has been indicated by $\phi_a(\vec{x})$. For an estimate of this term one can take the the Taylor expansion of G_1 , (5), and finds

$$\begin{aligned} |\phi_a(\vec{x})| &= \left| \frac{-1}{4\pi} \int_{r' \geq R} d^3x' (\vec{v}(\vec{x}') \cdot \vec{\nabla}') \left(\frac{\vec{x} \cdot \vec{x}'}{r'^3} + \dots \right) \right| \\ &\leq \left| \int_R^\infty dr' r'^2 v_0 \frac{1}{r'^\epsilon} \left[\frac{4r}{r'^3} + O\left(\frac{1}{r'^4}\right) \right] \right| \approx 4rv_0 \frac{1}{\epsilon R^\epsilon}. \end{aligned} \quad (22)$$

Thus the contribution of the outer domain to the potential vanishes as $1/\epsilon R^\epsilon$, and the existence of $\phi(\vec{x})$ has been proved.

It should be proven that the negative gradient of ϕ (15) represents the curl-free part \vec{v}_l of \vec{v}

$$\vec{v}_l(\vec{x}) = -\vec{\nabla} \phi(\vec{x}) = \frac{-1}{4\pi} \int d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) \vec{\nabla} G_1(\vec{x}, \vec{x}'). \quad (23)$$

Since \vec{v}_l is calculated from a potential curl \vec{v}_l is zero.

Now it should be shown that \vec{v}_l has the same sources as \vec{v}

$$\vec{\nabla} \cdot \vec{v}_l = \frac{-1}{4\pi} \int d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) \Delta G_1(\vec{x}, \vec{x}') = \vec{\nabla} \cdot \vec{v}. \quad (24)$$

Here we used the property (7) of the Green's function. Both vector fields have indeed the same sources. In (23) one can replace $\vec{\nabla}G_1(\vec{x},\vec{x}')$ by $-\vec{\nabla}'G_0(\vec{x},\vec{x}')$ and one obtains the longitudinal vector field in a manner that is known from the potential-theory in electro- and magnetostatic

$$\vec{v}_l(\vec{x}) = \int d^3x' \rho(\vec{x}') \vec{\nabla}'G_0(\vec{x},\vec{x}'). \quad (25)$$

The proof for the vector potential goes along the same lines and is shifted to appendix A.

2. Proof of uniqueness

Now, the existence of the potentials (15) and (16) has been proven. In the last step of the proof, the uniqueness of the decomposition has to be shown.

We have decomposed the vector field \vec{v} in a source field \vec{v}_l and a vortex field \vec{v}_t , under the boundary condition that the total field $|\vec{v}|$ vanishes going to infinity. In order to reach uniqueness of the decomposition we demand that $|\vec{v}_l|$ and in consequence also $|\vec{v}_t|$ vanish going to infinity.

Assume two different decompositions of the vector field $\vec{v} = \vec{v}_l + \vec{v}_t = \vec{v}_l' + \vec{v}_t'$ and consider the differences of the source and vortex fields. Then the difference of the longitudinal vector field $\vec{v}_d = \vec{v}_l - \vec{v}_l'$ is a divergence- and curl-free field that vanishes at infinity. It can be derived from a scalar potential ϕ_d , which fulfills the Laplace-equation (harmonic function). The only solution for the potential allowed would be a constant (see also the argumentation for (45) and (46)). Thus the vector field \vec{v}_d has to be zero and the decomposition is unique.

IV. APPLICATIONS IN ELECTRODYNAMICS

A. Static fields

In electrodynamics the fundamental theorem of vector analysis is used especially (although not always mentioned) in magneto-statics for magnetic fields in matter¹⁵. We consider a permanent magnet, where no volumen current is present. There (in the Gaussian system) the magnetization $4\pi\vec{M}$ corresponds¹⁶ to the vector field \vec{v} in the decomposition theorem, the magnetic field $-\vec{H}$ to the longitudinal (irrotational) part \vec{v}_l and the magnetic

induction \vec{B} to the transversal (solenoidal) part \vec{v}_t . The source is given by $\rho_H = -\vec{\nabla} \cdot \vec{M}$ and the circulation by $\vec{j}_H = \vec{\nabla} \times \vec{M}$.

These fields are related by the material equation $4\pi\vec{M} = \vec{B} - \vec{H}$. This exactly corresponds to the decomposition theorem. For such a case the sources and vortices are near the surface of the magnetic body, since the magnetization inside is almost constant. In any case the sources and vortices are localized to a finite region and in consequence the corresponding source and vortex field decay asymptotically at least as $1/r^2$. The total vector field \vec{v} of the magnetization is zero outside the magnetic body.

A quite similar situation occurs in electrostatics in a medium with spontaneous polarization, where no free charges are present¹⁵. There the vector field \vec{v} corresponds to the polarization $4\pi\vec{P}$, the source field \vec{v}_l to the electrostatic field $-\vec{E}$ and the vortex field \vec{v}_t to the dielectric displacement field \vec{D} . Again the decomposition theorem corresponds to the material equation $4\pi\vec{P} = \vec{D} - \vec{E}$.

Even in electro- and magnetostatics configurations with slow decreasing fields exist. The electric field of an infinite straight wire, which bears an electric charge, decays as $\sim 1/\rho$, where ρ is the distance to the wire. If on the other hand the wire carries a current, then the magnetic field decays as $\sim 1/\rho$. In both cases a regularization is appropriate to get the potentials from finite integrals over the sources.

B. Time dependent fields: the electric field of an oscillating dipole

Periodically moved charge densities $\rho(\vec{x}, t) = \rho(\vec{x}) e^{-i\omega t}$ of frequency ω emit a radiation field of the same frequency. For simplicity we use the complex notation understanding the physical quantities (charge density, potential, fields) always as the real parts of the corresponding complex quantities. The radiation fields factorize in the same way as the sources $\vec{v}(\vec{x}, t) = \vec{v}(\vec{x}) e^{-i\omega t}$, where in $\vec{v}(\vec{x})$ the dependence on the frequency ω resp. wave number $k = \omega/c$ has been suppressed. A decomposition of the time independent vector field $\vec{v}(\vec{x})$ is possible, since the radiation field, or more precisely its long range part, decays as $1/r$ and thus fulfills clearly the conditions of the decomposition theorem.

If one starts from the assumption that the asymptotic behavior of the field has to be stronger than $1/r$, additional considerations are needed in order to proof the decomposition of the radiation fields¹⁷. Radiation fields, which decay asymptotically as $1/r$ are rarely con-

nected with the decomposition theorem. One reason might be that in most of the textbooks on electrodynamics the result of Blumenthal's proof are not mentioned and one gets the impression the decomposition theorem can only applied under additional conditions¹⁷ as they are found in radiation fields like e^{ikr}/r . The peculiarity of these cases is, that one does not need the regularization term, although one has a field of $O(1/r)$.

Strictly speaking the conditions of the theorem are not fulfilled if the vector field has singularities due to point sources. This also holds for the radiation fields considered. However the integration over the sources in (15) and (16) remain finite. The only consequence, in cases where a regularization is necessary, is that the regularization point has to be different from the singular points due to the source.

The electric radiation field $\vec{E}(\vec{x}) \equiv \vec{v}(\vec{x})$ of an oscillating point dipole $\vec{p}(t) = \vec{p}e^{-i\omega t}$ reads¹⁸

$$\vec{v}(\vec{x}) = \frac{e^{ikr}}{r} \left\{ k^2 \vec{e}_r \times (\vec{p} \times \vec{e}_r) + \frac{1}{r^2} (1 - ikr) \left[3(\vec{p} \cdot \vec{e}_r) \vec{e}_r - \vec{p} \right] \right\}. \quad (26)$$

$\vec{e}_r = \vec{x}/r$ is the unit vector in the direction of \vec{x} and $\vec{v}(\vec{x})$ is the spatial part of the electric field. For $k = \omega/c = 0$, one obtains of course the static dipole field. Let us first calculate the source and vortex density

$$\rho_H(\vec{x}) = \frac{\vec{\nabla} \cdot \vec{v}}{4\pi} = e^{ikr} (1 - ikr) \rho_p(\vec{x}) \hat{=} \rho_p(\vec{x}) \quad \rho_p(\vec{x}) = -\vec{p} \cdot \vec{\nabla} \delta(\vec{x}) \quad (27)$$

$$\vec{j}_H(\vec{x}) = \frac{\vec{\nabla} \times \vec{v}}{4\pi} = -\frac{e^{ikr}}{4\pi} \frac{k^2}{r^2} (1 - ikr) (\vec{e}_r \times \vec{p}) = \frac{ik}{4\pi} \vec{B}(\vec{x}). \quad (28)$$

$\rho_p(\vec{x})$ is the localized charge density of the static dipole¹⁹, whereas the vortex density is extended in the whole domain decreasing for $r \rightarrow \infty$ as the radiation field with $1/r$. It can be identified with the spatial part of the magnetic radiation field¹⁸ \vec{B} apart from a factor, as expected from Faraday's law of induction. Surprisingly the wave number dependence in $\rho_H(\vec{x})$, which in $\vec{j}_H(\vec{x})$ comes from the retardation, drops out. This asymmetry has already been discussed by Brill and Goodman²⁰. Hence the scalar potential is given by

$$\phi_H(\vec{x}) = \int d^3x' \frac{\rho_H(\vec{x}')}{|\vec{x}' - \vec{x}|} = \frac{\vec{p} \cdot \vec{e}_r}{r^2} = \phi_C(\vec{x}). \quad (29)$$

Multiplying by the factor $e^{-i\omega t}$ one obtains the quasi-static (acausal) dipole potential $\phi_C(\vec{x}, t)$ as it is known using the Coulomb gauge²¹. One obtains the spatial part of the scalar potential in Coulomb gauge²¹, which is the static (acausal) dipole potential. From that it is clear that the longitudinal decomposed vector field \vec{v}_l is the quasi-static electric field of a point dipole

$$\vec{v}_l(\vec{x}) = -\vec{\nabla} \phi_H(\vec{x}) = \left[-\vec{p} + 3(\vec{p} \cdot \vec{e}_r) \vec{e}_r \right] \frac{1}{r^3} \quad (30)$$

and does not contribute to the electromagnetic radiation, which is pure transversal. The decomposition is finally shown by calculating the transversal part $\vec{v}_t = \vec{\nabla} \times \vec{A}_H$ according to the theorem from of the vector potential

$$\vec{A}_H(\vec{x}) = \int d^3x' \frac{\vec{j}(\vec{x}')}{|\vec{x}' - \vec{x}|} = k^2 \vec{p} \times \vec{e}_r \left[\frac{e^{ikr}}{ikr} + \frac{1}{k^2 r^2} (e^{ikr} - 1) \right] = \frac{i}{k} \vec{B}(\vec{x}) + \frac{\vec{e}_r}{r^2} \times \vec{p}. \quad (31)$$

In electrodynamics one never defines a vector potential for the electric field, but it is known from the Ampère-Maxwell-equation that the electric field can be calculated via the curl of \vec{B} . The longitudinal part of \vec{v}_l is removed by the second term of \vec{A}_H , a quasistatic vector field. Note that this is **not** the vector potential \vec{A}_C known from calculating the electric and magnetic fields in the Coulomb gauge

$$\begin{aligned} \vec{A}_C(\vec{x}) &= -\frac{e^{ikr}}{r} \left\{ ik \vec{e}_r \times (\vec{p} \times \vec{e}_r) - [\vec{p} - 3(\vec{e}_r \cdot \vec{p})\vec{e}_r] \left[\frac{1}{r} + \frac{i}{kr^2} (1 - e^{-ikr}) \right] \right\} \\ &= \frac{1}{ik} (\vec{E}(\vec{x}) + \nabla \phi_C(\vec{x})). \end{aligned} \quad (32)$$

Thus with the transverse field

$$\begin{aligned} \vec{v}_t(\vec{x}) &= \frac{e^{ikr}}{r} \left\{ k^2 \vec{e}_r \times (\vec{p} \times \vec{e}_r) + \frac{1}{r^2} (1 - ikr) \left[3(\vec{p} \cdot \vec{e}_r) \vec{e}_r - \vec{p} \right] \right\} - \frac{1}{r^2} [3(\vec{p} \cdot \vec{e}_r) \vec{e}_r - \vec{p}] \\ &= \vec{E}(\vec{x}) - \vec{v}_l(\vec{x}). \end{aligned} \quad (33)$$

the causal character of the total electric radiation field $\vec{v}(\vec{x})$ is restored^{22,23}.

The same decomposition may be done for the magnetic radiation field, which however is trivial since the field is only transversal (see (28)). The vector potential fulfills $\vec{\nabla} \cdot \vec{A}_H = 0$ and $\vec{\nabla} \times \vec{A}_H = \vec{B}$, the same conditions as for the vector potential \vec{A}_C in the Coloumb gauge. We obtain indeed $\vec{A}_H(\vec{x}) = \vec{A}_C(\vec{x})$. The vortex density of the magnetic field is apart from a factor given by the same expression as the electric field $\vec{E}(\vec{x})$ (26) of the electric dipole radiation

$$\vec{j}_H(\vec{x}) = \frac{1}{4\pi} \vec{\nabla} \times \vec{B}(\vec{x}). \quad (34)$$

Thus all the fields, the vector potential $\vec{A}_H(\vec{x})$, the vortex field $\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}_H(\vec{x})$ and the vortex density $\vec{j}_H(\vec{x}) = \vec{\nabla} \times \vec{B}(\vec{x})/4\pi$ decay asymptotically as $1/r$. This is a consequence of retardation. We also note that the last term in (16), the regularization term, which guaranties the convergence for a weak decrease of the field as $1/r$, is not necessary in this case. The integrals converge even without this term¹⁷. This also applies for other fields like

$\vec{v}(\vec{x}) = \vec{p}/r$. On the contrary, for a vector field like $\vec{v}(\vec{x}) = \vec{e}_r/r$ the regularization term is necessary for reaching convergence, but the regularization point \vec{x}_0 has to be different from zero. Then we get for the potential

$$\phi(\vec{x}) = \ln r_0 - \ln r. \quad (35)$$

One may be surprised that all calculations for the radiation field could be performed with a regularization on a lower level (G_0 instead of G_1 etc) than expected according to the decay of the vector field. One reason lies in the symmetries of the sources and circulations (see App. C).

V. DIVERGING VECTOR FIELDS

A. Supplement to the fundamental theorem of vector analysis

As already mentioned, the fundamental theorem of vector analysis can be applied to asymptotically sublinearly **diverging** vector fields, if one inserts the faster decaying Green's function G_2 (6) into (15) and (16) for the computation of ϕ and \vec{A} .

Let $\vec{v}(\vec{x})$ be an everywhere continuous differentiable vector field with the asymptotic behavior $\lim_{r \rightarrow \infty} v(r)/r^{1-\epsilon} < \infty$, with $\epsilon > 0$, then the decomposition

$$\vec{v}(\vec{x}) = \vec{v}_l + \vec{v}_t + \vec{v}_c = -\vec{\nabla}\tilde{\phi}(\vec{x}) + \vec{\nabla} \times \vec{A}(\vec{x}) + \vec{v}(\vec{x}_0) \quad (36)$$

with

$$\tilde{\phi}(\vec{x}) = \frac{1}{4\pi} \int d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) \quad (37)$$

$$\vec{A}(\vec{x}) = \frac{1}{4\pi} \int d^3x' (\vec{\nabla}' \times \vec{v}(\vec{x}')) G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) \quad (38)$$

is unique apart from a constant vector field.

Remarks:

- The vector potential \vec{A} by its definition is purely transversal, $\vec{\nabla} \cdot \vec{A} = 0$ (see proof below).
- The regularization of the Green's function at a point \vec{x}_0 is responsible for the finiteness of the integrals (37) and (38). The point can be chosen arbitrarily.

- Curl- and divergence-free (harmonic) fields \vec{v}_h can be added to \vec{v}_l if they are subtracted from \vec{v}_t without affecting \vec{v} .
- Harmonic fields with the exception of constant vector fields \vec{v}_c can be suppressed if one demands that \vec{v}_l asymptotically diverge weaker as linearly. This is shown in section V A 2.
- If \vec{v}_l and \vec{v}_t are calculated with (37) and (38), then one obtains the value of the constant vector $\vec{v}_c = \vec{v}(\vec{x}_0)$ depending on the arbitrary regularization point.
- If the vector field approaches zero slower than any power law or if it diverges logarithmically (as it is the case in Blumenthal's formulation of the theorem), or if it increases sublinearly, then the faster converging Green's function G_2 has to be applied in ϕ and \vec{A} . The price one has to pay for this weaker requirements on the vector field \vec{v} is the loss of the rigorous uniqueness of the decomposition.
- If one uses the regularized Green's function G_2 for the case where the vector field \vec{v} decreases stronger, one recovers the unique decomposition of the fundamental theorem (14), since all integrals coming from the regularization terms are finite.
- In the special case of the theorem where \vec{v} approaches zero at infinity weaker as any power of $1/r$ (the case $\epsilon = 1$), then v_l and v_t may diverge logarithmically although the sum of the two parts decays to zero⁶.

1. Proof of the supplementary theorem

At first one has to show the existence of $\tilde{\phi}$ and $\vec{\tilde{A}}$, (37) and (38). Concerning the potentials $\tilde{\phi}$ and $\vec{\tilde{A}}$, their integrand has the same asymptotic decay governed by $\vec{v}(\vec{x}')G_2(\vec{x}') \sim 1/r^{2+\epsilon}$, as ϕ and \vec{A} in the former proof for the fundamental theorem. $G_2(\vec{x}, \vec{x}')$ (6) has compared to $G_1(\vec{x}, \vec{x}')$ an additional singular term at $\vec{x}_0 = 0$. We have to prove that the contribution of this singularity to $\tilde{\phi}$ (and $\vec{\tilde{A}}$) is finite. For this purpose we integrate over a small sphere of radius $\eta \rightarrow 0$ around zero ($\xi' = \cos \vartheta'$)

$$\tilde{\phi}_\eta(\vec{x}) = \frac{-1}{4\pi} \int_{S_\eta} d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) \frac{\vec{x} \cdot \vec{x}'}{r'^3} = -2\pi\rho(0) r \int_0^\eta dr' \int_{-1}^1 d\xi' \xi' = 0.$$

Now we can be sure that $\tilde{\phi}(\vec{x})$ and $\vec{A}(\vec{x})$ exist. Starting from (15), we can reformulate all equations up to (A6) by replacing G_i by G_{i+1} .

Before the decomposed vector fields are computed, one should compare the scalar potentials (16) with (38). Because of the use of G_2 in $\tilde{\phi}$ these both potentials differ in linear function in \vec{x} . This applies even to the difference between \vec{A} and \vec{A} and has the consequence that \vec{v}_l and \vec{v}_l are indeterminate by a constant vector. Starting with (37) we build \vec{v}_l as the negative gradient of $\tilde{\phi}$ and check if \vec{v}_l has the same sources as \vec{v}

$$\begin{aligned}\vec{v}_l(\vec{x}) &= -\vec{\nabla}\tilde{\phi}(\vec{x}) = -\frac{1}{4\pi} \int d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) \vec{\nabla} G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) \\ \vec{\nabla} \cdot \vec{v}_l(\vec{x}) &= -\frac{1}{4\pi} \int d^3x' (\vec{\nabla}' \cdot \vec{v}(\vec{x}')) \Delta G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = \vec{\nabla} \cdot \vec{v}(\vec{x}).\end{aligned}\tag{39}$$

We find that everything holds as expected. Now we rewrite \vec{v}_l by using (13) and performing a partial integration

$$\vec{v}_l(\vec{x}) = \int d^3x' \rho(\vec{x}') \vec{\nabla}' \left(\frac{1}{|\vec{x}' - \vec{x}|} - \frac{1}{|\vec{x}' - \vec{x}_0|} \right).\tag{40}$$

As can be seen from (40) one gets $\vec{v}_l(\vec{x}_0) = 0$. If one compares $\vec{v}_l(\vec{x})$ computed with G_0 in (25) one sees that the additional term of G_1 subtracts a (divergent) constant field from the first term to hold $\vec{v}_l(\vec{x})$ finite. This divergence- and curl-free vector field does not contribute to the source density.

A partial integration in the vector potential (38) yields to (see (A1))

$$\vec{A}(\vec{x}) = \frac{1}{4\pi} \int d^3x' \vec{v}(\vec{x}') \times \vec{\nabla}' G_2(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0).\tag{41}$$

As already mentioned, differ \vec{A} and \vec{A} by a vector linearly in \vec{x} . Even $\vec{A}(\vec{x})$ is purely transversal (use (13) and compare the result with (A2))

$$\vec{\nabla} \cdot \vec{A}(\vec{x}) = \frac{-1}{4\pi} \int d^3x' (\vec{v}(\vec{x}') \times \vec{\nabla}') \cdot \vec{\nabla}' G_1(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) = 0\tag{42}$$

Now the vortex field is calculated from the vector potential by taking its curl and transform $\vec{\nabla} G_2$ to $-\vec{\nabla}' G_1$

$$\vec{v}_t(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) = \frac{1}{4\pi} \int d^3x' (\vec{\nabla}' \times \vec{v}(\vec{x}')) \times \vec{\nabla}' G_1(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0).\tag{43}$$

Analogous in the case of the irrotational vector \vec{v}_l also (43) contains in addition to (A3) a constant vector field effecting that $\vec{v}_t(\vec{x}_0) = 0$. Now it is shown that the vortices of \vec{v}_t are the same as for \vec{v} . Inserting G_2 into (A4) one obtains that $\vec{\nabla} \times \vec{v}_t = \vec{\nabla} \times \vec{v}$.

Our interest is directed to the sum $\vec{v}_l + \vec{v}_t$ since both fields have divergence- and curl-free constant vectors. They inhibit that \vec{v} is the sum of $\vec{v}_l + \vec{v}_t$ as the following calculation shows. If one replace in (A6) G_2 by G_3 and takes $\vec{x}_0 \neq 0$ one obtains

$$\vec{v}_l(\vec{x}) + \vec{v}_t(\vec{x}) = \frac{-1}{4\pi} \int d^3x' \Delta(\vec{v}(\vec{x}') G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0)) = \vec{v}(\vec{x}) - \vec{v}(\vec{x}_0). \quad (44)$$

Now we identify the constant vector $\vec{v}(\vec{x}_0)$ with \vec{v}_c in (36). Thus \vec{v}_c , \vec{v}_l and \vec{v}_t depend all on \vec{x}_0 .

2. On the uniqueness in the case of an increasing vector field

Let us start from two different solutions of the decomposition $\vec{v}_{l,t}$ and $\vec{v}'_{l,t}$ with the same sources and vortices respectively. Then the difference $\vec{v}_d = \vec{v}_l - \vec{v}'_l$ is a irrotational solenoidal vector field.

This vector field can be written as the gradient of a scalar potential ϕ_d that fulfills the Laplace equation $\Delta\phi_d = 0$ in the whole space. Its most general solution in spherical coordinates reads:

$$\phi_d(r, \vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (\alpha_{lm} r^l + \beta_{lm} r^{-l-1}) Y_{lm}(\vartheta, \varphi) \quad (45)$$

where α_{lm} and β_{lm} are coefficients which allow the solution to fulfill the boundary conditions and Y_{lm} are the spherical harmonics. All β_{lm} vanish because the zero-point is contained within the domain and the solution should be regular. We now calculate the radial harmonic flux of the vector field

$$\vec{v}_d \cdot \vec{e}_r = v_{dr} = -\frac{\partial\phi_d}{\partial r} = -\sum_{l=1}^{\infty} \sum_{m=-l}^l \alpha_{lm} l r^{l-1} Y_{lm}(\vartheta, \varphi). \quad (46)$$

When the distance r goes to ∞ and one notes that $v_d \sim r^{1-\epsilon}$ with $\epsilon > 0$, then all coefficients with $l-1 > 1-\epsilon$ have to vanish, otherwise this would lead to a stronger divergence of v_{dr} . Thus only the terms with $l=0$ und $l=1$ remain. Therefore the solution reads:

$$\phi_d(r, \vartheta, \varphi) = \alpha_{00} Y_{00} + \sum_{m=-1}^1 \alpha_{1m} Y_{1m}(\vartheta, \varphi) r = \frac{\alpha_{00}}{\sqrt{4\pi}} - \vec{w} \cdot \vec{x}, \quad (47)$$

and we obtain

$$\vec{v}_d(\vec{x}) = -\vec{\nabla}\phi_d = \vec{w}. \quad (48)$$

Hence the field $\vec{v}_d = \vec{w}$ is unique up to a constant vector. The choice of the regularization point influences the constant vector of \vec{v}_l only.

3. Comment on the application of the supplementary theorem

We are not aware of a an analytically calculable physical example in this case, however for numerical calculations the knowledge of the validity of such a theorem is important. There are cases where one does not know always the exact asymptotic behavior of a vector field. We have already seen that in the case of electromagnetic radiation because of the (symmetry) properties of the field one could decompose the field with the Green function G_0 although generally G_1 should be necessary. A few statements on the influence of the symmetry are made in the appendix C.

Generally we want to point out that the application of the regularization schema is not only restricted to the decomposition of a vector field, but may also be important for other problems where the solution of a poisson equation (scalar or vectorial) is used.

If one considers sources that remain finite at infinity, the vector field belonging to this sources should diverge linearly or stronger. Within our schema one then has tu use the next step in the regularization procedure. There the vector field has to be computed with G_2 and is determined only up to a vector field linear in \vec{x} .

4. Stronger diverging vector fields

We have already seen that asymptotically strong decaying Green's functions G_i with $i \geq 3$ cannot be treated in the same manner a those for $i \leq 2$. We restrict ourselves to $i = 3$ what means that the vector field may increase less than quadratically. In this case the potentials are given by (see (12))

$$\begin{aligned}\bar{\phi}_3(\vec{x}) &= \int d^3x' \rho(\vec{x}') G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) + 2\pi \frac{\rho(\vec{x}_0)}{3} r^2 \\ \vec{A}_3(\vec{x}) &= \int d^3x' \vec{j}(\vec{x}') G_3(\vec{x} - \vec{x}_0, \vec{x}' - \vec{x}_0) + 2\pi \frac{\vec{j}(\vec{x}_0)}{3} r^2.\end{aligned}\tag{49}$$

The last term of ϕ_3 and \vec{A}_3 cancels the contribution to the inhomogeneity caused by G_3 . Now, \vec{v} can again be decomposed in \vec{v}_l and \vec{v}_t except for a linear vector field that depends on the regularization point \vec{x}_0 .

Remarks:

- Besides our statements to the regularization of weak diverging vector fields high symmetric vector fields can be decomposed by the method shown here even if they diverge

asymptotic region	Exponent	Green's function	$v_l(r \rightarrow \infty)$	Unique up to
$v \sim r^{1-\epsilon}$	$0 < \epsilon \leq 1$	$G_2(\vec{x}, \vec{x}')$	$v_l \sim r^{1-\epsilon}$	\vec{w}
$v \sim 1/r^\epsilon$	$0 < \epsilon \leq 1$	$G_1(\vec{x}, \vec{x}')$	$v_l(\infty) = 0$	$\vec{w} = 0$
$v \sim 1/r^{1+\epsilon}$	$0 < \epsilon$	$G_0(\vec{x}, \vec{x}')$	$v_l(\infty) = 0$	$\vec{w} = 0$

TABLE I. Different cases of a vector fields $\vec{v}(\vec{x})$, which decay asymptotically to zero or increase sublinearly (first and second column), can be decomposed into longitudinal (irrotational, curl-free) \vec{v}_l and transversal (solenoidal, divergence-free) parts besides a constant vector field \vec{w} . In order to cover all cases one has to introduce regularized Green's functions (see (5)) (third column) respectively. Also shown is the asymptotic condition on the longitudinal field (fourth column). We also indicate the extent of the uniqueness of the decomposition (fifth column).

stronger than assumed so far. This can be seen from (B4) for $i = 0$ (and $i = 1$), where the additional terms to G_1 in G_2 cancel the contributions to S_0 (and S_1) for $r' > r$.

- Simple examples for this feature are $\vec{v} = \vec{a}r^\alpha$ and $\alpha > -1$ or $\vec{v} = \vec{e}_r r^\alpha$.

VI. CONCLUSION

We have presented a proof of the fundamental theorem of vector analysis (Helmholtz' decomposition theorem) for vector fields decaying weakly and extended to even sublinearly diverging vector fields. Contrary to the original proof⁶ we can distinguish between different cases. Our results are summarized in Tab. I. Note however that not only the decay of the vector field is important for introducing a regularization but also its symmetry. This extends the presentations of this theorem given usually in textbook on electrodynamics. Especially the case of weakly decaying fields has been discussed in the physical literature in the context of electromagnetic radiation fields.

Considering the validity of Helmholtz' decomposition theorem there is no doubt that the theorem can be applied quite generally to electromagnetic fields either static or dynamic. This was demonstrated by explicit examples.

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Appendix A: Existence of the vector potential and its transverse vector field

For the vector potential (16) we have the problem that we do not know the asymptotic behavior of the vortex density. Therefore we show the existence of the integral (16) in the same way as in subsection III 1 and obtain after a partial integration

$$\vec{A}(\vec{x}) = \frac{1}{4\pi} \int d^3x' \vec{v}(\vec{x}') \times \vec{\nabla}' G_1(\vec{x}, \vec{x}'). \quad (\text{A1})$$

This vector potential $\vec{A}(\vec{x})$ turns out to be a purely transversal vector potential for which the divergence vanishes. In order to show this we use (13) in (A1)

$$\vec{\nabla} \cdot \vec{A}(\vec{x}) = \frac{-1}{4\pi} \int d^3x' (\vec{v}(\vec{x}') \times \vec{\nabla}') \cdot \vec{\nabla}' G_0(\vec{x}, \vec{x}') = 0. \quad (\text{A2})$$

The fundamental theorem of vector analysis states that the solenoidal part of \vec{v} field is given by (see (A1))

$$\vec{v}_t(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) = \frac{1}{4\pi} \int d^3x' \vec{\nabla} \times (\vec{\nabla}' \times \vec{v}(\vec{x}')) G_1(\vec{x}, \vec{x}'). \quad (\text{A3})$$

Since \vec{v}_t is calculated from the vector potential $\text{div } \vec{v}_t$ is zero.

Now one has to show that the vortices of \vec{v}_t are the same as those of the given vector field \vec{v} . Therefore we calculate the curl of (A3) and use the identity $(\vec{\nabla} \times \vec{\nabla}') \vec{v} = \vec{\nabla}(\vec{\nabla}' \cdot \vec{v}) - \Delta \vec{v}$

$$\begin{aligned} \vec{\nabla} \times \vec{v}_t(\vec{x}) &= \frac{1}{4\pi} \int d^3x' \vec{\nabla} \times [\vec{\nabla} \times (\vec{\nabla}' \times \vec{v}(\vec{x}')) G_1(\vec{x}, \vec{x}')] \\ &= \frac{-1}{4\pi} \int d^3x' \Delta G_1(\vec{x}, \vec{x}') (\vec{\nabla}' \times \vec{v}(\vec{x}')) = \vec{\nabla} \times \vec{v}(\vec{x}). \end{aligned} \quad (\text{A4})$$

\vec{v}_t is a pure solenoidal field and its vortices of \vec{v}_t are the same as those of \vec{v} .

In a final step it is shown the sum of the irrotational and solenoidal field $\vec{v}_l + \vec{v}_t = \vec{v}$ equals the given vector field. For this reason we reshape $\phi(\vec{x})$ (21) and $\vec{A}(\vec{x})$ (A1) by replacing $\vec{\nabla}' G_1(\vec{x}, \vec{x}')$ with $-\vec{\nabla}' G_2(\vec{x}, \vec{x}')$ according to (13)

$$\begin{aligned} \phi(\vec{x}) &= \frac{1}{4\pi} \int d^3x' \vec{\nabla} \cdot \vec{v}(\vec{x}') G_2(\vec{x}, \vec{x}') \\ \vec{A}(\vec{x}) &= \frac{1}{4\pi} \int d^3x' \vec{\nabla} \times \vec{v}(\vec{x}') G_2(\vec{x}, \vec{x}'). \end{aligned} \quad (\text{A5})$$

Now the negative gradient of $\phi(\vec{x})$ is added to the curl of $\vec{A}(\vec{x})$ and the identity $(\vec{\nabla} \times \vec{\nabla})\vec{v} = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \Delta\vec{v}$ is used

$$\begin{aligned}\vec{v}_l(\vec{x}) + \vec{v}_t(\vec{x}) &= \frac{-1}{4\pi} \int d^3x' \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v}(\vec{x}')) - \vec{\nabla} \times (\vec{\nabla} \times \vec{v}(\vec{x}')) \right] G_2(\vec{x}, \vec{x}') \\ &= \frac{-1}{4\pi} \int d^3x' \Delta(\vec{v}(\vec{x}')) G_2(\vec{x}, \vec{x}') = \vec{v}(\vec{x}).\end{aligned}\quad (\text{A6})$$

Appendix B: Example of a diverging vector field

We want to study the following vector field

$$\vec{v} = \vec{a} \times (e_r \times \vec{a})\sqrt{r} \quad (\text{B1})$$

where \vec{a} is a constant vector. \vec{v} diverges as $\sim \sqrt{r}$. It seems to be more convenient to determine first sources and vortices and then to calculate the fields belonging to these

$$\rho(\vec{x}) = \frac{1}{4\pi} [3a^2 + (\vec{a} \cdot \vec{e}_r)^2] \frac{1}{2\sqrt{r}} \quad \vec{j}(\vec{x}) = \frac{1}{4\pi} (\vec{a} \cdot \vec{e}_r) \vec{e}_r \times \vec{a} \frac{1}{2\sqrt{r}}. \quad (\text{B2})$$

To make the computation of the potentials as simple as possible we use the regularization point $\vec{x}_0 = 0$. Then we get ϕ from (37) as follows

$$\phi(\vec{x}) = \frac{1}{4\pi} \int d^3x' \frac{1}{2\sqrt{r'}} [3a^2 + (\vec{a} \cdot \vec{e}_{r'})^2] G_2(\vec{x}, \vec{x}') \quad (\text{B3})$$

In the next step we introduce spherical coordinates and we fix the primed coordinate system by the unprimed vector \vec{x} : $\vec{e}_{z'} = \vec{e}_r$ and perform the integration over the azimuth φ' ($\xi' = \cos\vartheta'$). This leads to a replacement of $\sin\varphi' \cos\varphi'$ by zero and $\cos^2\varphi' = \sin^2\varphi'$ by $1/2$

$$\begin{aligned}\vec{a} \cdot \vec{e}_{r'} &= a_{x'} \sqrt{1 - \xi'^2} \cos\varphi' + a_{y'} \sqrt{1 - \xi'^2} \sin\varphi' + a_{z'} \xi' \\ (\vec{a} \cdot \vec{e}_{r'})^2 &= a^2 \frac{1}{2} (1 - \xi'^2) + (\vec{a} \cdot \vec{e}_r)^2 \frac{1}{2} (3\xi'^2 - 1)\end{aligned}$$

For the calculation of the angular integral of ϕ one needs to evaluate this two surface integrals ($d\Omega' = d\xi' d\varphi'$)

$$S_i(r, r') = \frac{1}{4\pi} \int d\Omega' \xi'^i G_2(\vec{x}, \vec{x}') \quad (\text{B4})$$

$$S_0(r, r') = \left(\frac{1}{r} - \frac{1}{r'} \right) \theta(r - r') \quad (\text{B5})$$

$$S_2(r, r') = \frac{1}{3} S_0 + \frac{2}{15} \left\{ \frac{r'^2}{r^3} \theta(r - r') + \frac{r^2}{r'^3} \theta(r' - r) \right\} \quad (\text{B6})$$

Now we get for the scalar potential

$$\begin{aligned}\phi(\vec{x}) &= \int_0^\infty dr' \sqrt{r'}^3 \frac{1}{4} \left\{ [7a^2 - (\vec{a} \cdot \vec{e}_r)^2] S_0(r, r') - [a^2 - 3(\vec{a} \cdot \vec{e}_r)^2] S_2(r, r') \right\} \\ &= \frac{1}{9} [-7a^2 + 2(\vec{a} \cdot \vec{e}_r)^2] \sqrt{r}^3\end{aligned}\quad (\text{B7})$$

The analogous calculation for the vector potential yields

$$\vec{A}(\vec{x}) = \frac{2}{9} \vec{a} \times (\vec{e}_r \times \vec{a}) \sqrt{r}^3. \quad (\text{B8})$$

In the last step, the calculation of the decomposed vector fields, we get

$$\vec{v}_l(\vec{x}) = -\vec{\nabla} \phi(\vec{x}) = -\frac{1}{9} \left\{ [7a^2 - (\vec{a} \cdot \vec{e}_r)^2] \vec{e}_r + 4(\vec{a} \cdot \vec{e}_r)^2 \vec{e}_r \right\} \sqrt{r} \quad (\text{B9})$$

$$\vec{v}_t(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) = \frac{1}{9} \left\{ 2\vec{a} \times (\vec{e}_r \times \vec{a}) - 4(\vec{a} \cdot \vec{e}_r) \vec{a} + (\vec{a} \cdot \vec{e}_r) \vec{e}_r \times (\vec{a} \times \vec{e}_r) \right\} \sqrt{r}. \quad (\text{B10})$$

Thus we have demonstrated that sublinearly divergent vector fields can be decomposed in its irrotational and solenoidal components, both diverging as $\sim \sqrt{r}$. Since $\vec{v}(\vec{x}_0 = 0)$ vanishes, it is indeed $\vec{v}_l + \vec{v}_t = \vec{v}$.

Appendix C: On the influence of symmetry

In the preceding example we used (B4) for the computation of $\phi(\vec{x})$. The θ -functions in (B4) cuts all diverging contributions for $r' \rightarrow \infty$ in (B7). Similarly a Helmholtz vector potential $\vec{A}(\vec{x})$ calculated for a circulation with the symmetry $\vec{j}(\vec{x}) = f(r) \vec{p}$ exists for arbitrary diverging $f(r)$.

In the case of the electromagnetic example in Sec. IV B the Helmholtz vector potential $\vec{A}_H(\vec{x})$ of the electric radiation field (31) the same effect happens. The required surface integral is calculated for a current (28) of the form $\vec{j}_H = f(r) \vec{e}_r \times \vec{p}$ and reads

$$S_1^0(r, r') = \frac{1}{4\pi} \int d\Omega' \xi' \frac{\xi'}{|\vec{x} - \vec{x}'|} \frac{1}{3} \left[\frac{r'}{r^2} \theta(r - r') + \frac{r}{r'^2} \theta(r' - r) \right].$$

The integral converges for $r' \rightarrow \infty$ with $1/r'^2$ which is stronger as expected from the behavior of G_0 alone, but one gets the same result for the surface integral if one uses G_1 instead of G_0 . This makes clear that the type of regularization necessary depends on the symmetry of the vector field \vec{v} considered and why we did not need G_1 in the case of electromagnetic

radiation.

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- ² Stokes G. On the dynamical theory of diffraction, *Trans. Cambridge Phil. Soc.* **9**, S. 1, Compl. Works vol. II, see p. 10, item 8
- ³ Already Stokes² has 1845 performed such an analysis of the movement of a liquid. This is more explicitly explained in Lamb's *Hydrodynamics* Dover Publ. Reprint 1932 chapterl III item 30 p. 31.
- ⁴ See the review by Bhatia H, Norgard G, Pascucci V and Bremer P T 2013 The Helmholtz-Hodge Decomposition - A Survey *IEEE Transactions on Visualization and Computer Graphics* **19** 1386-1404
- ⁵ Abraham M 1918 *Theorie der Elektrizität*, vol. 1, 5. Aufl. G.G. Teubner Leipzig, 1. Aufl. 1894
- ⁶ Blumenthal O 1905 Über die Zerlegung unendlicher Vektorfelder *Math. Ann.* **61** 235
- ⁷ See Ref.⁶ p. 236
- ⁸ Sommerfeld A 1970 *Vorlesungen über Theoretische Physik* Bd. II, 6. Aufl. Akademische Verlagsgesellschaft Leipzig and 1950 *Mechanics of Deformable Bodies: Lectures on Theoretical Physics*, Vol. 2, Academic Press New York
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- ¹⁰ Tran-Cong Ton 1993 On Helmholtz's Decomposition Theorem and Poisson's Equation with an Infinite Domain *Quarterly of applied mathematics* **51** 23-35
- ¹¹ Neudert M and Wolf von Wahl 2001 Asymtotic Behavior of the Div-Curl Problem in Exterior Domains *Advances in Differential Equations* **6** 1347-1376
- ¹² See for an exception Petrascheck D und Schwabl F 2015 *Elektrodynamik* Springer Spektrum Heidelberg, see p. 221, remark 4 there.
- ¹³ For an exception see Großmann S 1981 *Mathematischer Einführungskurs für Physiker* Teubner Studienbücher Physik Stuttgart

- ¹⁴ There were several items to clarify for time dependent vector fields, especially the question of retardation, its connection to causality and the choice of gauge.
- ¹⁵ Miller B P 1984 Interpretations from Helmholtz' theorem in classical electromagnetism *Am. J. Phys.* **52** (10) 948-950
- ¹⁶ Petrascheck D und Schwabl F 2015 *Elektrodynamik* Springer Spektrum Heidelberg, see p. 221
- ¹⁷ Stewart A M 2014 Does the Helmholtz theorem of vector decomposition apply to the wave fields of electromagnetic radiation? *Phys. Scr.* **89** 065502
- ¹⁸ see¹⁶ (8.4.5) and (8.4.6) p. 294. The time dependence is in this case contained in the Fourier factor $e^{-i\omega t}$
- ¹⁹ The factor $e^{ikr}(1 - ikr)$ is a consequence of the singular structure of ρ_p and can be replaced by 1 in all calculations.
- ²⁰ Brill O L and Goodman B 1987 Causality in the Coulomb Gauge *Am. J. Physics* **35** 832-837
- ²¹ The appearance of the quasistatic potentials has led to a discussion on the validity of causality in the case of Coulomb gauge. However it was recognized earlier²⁰ and confirmed later²², that the physical quantities are causal and the decomposition is valid also for time dependent (retarded) fields.
- ²² Rohrlich F 2004 The validity of the Helmholtz theorem *Am. J. Physics* **72** 412-413
- ²³ Jackson J D 2010 Comment on 'Maxwell equations and redundant gauge degree of freedom' *Eur. J. Phys.* **31** L79